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# Dynamic Linearization and Ω-Observability of Nonlinear Systems

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A construction of a state estimator is given for a class of nonlinear systems $S$ in a region of approximation $\Omega$ about the origin. For deterministic systems, the estimator is exact in $\Omega$ if the dynamically linearized approximation $S^1$ to $S$ in $\Omega$ passes a rank test. The structure of the estimator is nonlinear and depends on "synthetic observations" which are formed from powers and products of the measurements. The concept of the "synthetic observations" must be modified when the measurements are corrupted by noise.

## 1. DETERMINISTIC NONLINEAR SYSTEMS

In this section some results on the estimation of the states of a class of deterministic nonlinear systems are given. These results apply within the context of Dynamic Linearization [1, 2] so that the state estimators are only valid approximations to the true state values in some region specified by the Dynamic Linearization procedure. With no loss of generality the region of approximation from here on will be taken about the origin. Thus the system

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) \tag{1.1}$$

expanded about the origin becomes

$$\dot{\mathbf{z}} = J\mathbf{z} + \mathbf{r}(\mathbf{z}), \tag{1.2}$$

where $J_{ij} = \partial f_i / \partial z_j$ and $\mathbf{r}(\mathbf{z})$ contains higher order products and powers of $\mathbf{z}$ i.e. $z_i z_j \cdots z_k$. Dynamic Linearization considers powers and products of $\mathbf{z}$ up to a specified order as extra states and defines a region of approximation $\Omega$ about the origin in which still higher order powers and products of $\mathbf{z}$ are considered negligible. The retained higher order states $z_i z_j \cdots z_k$ together with the original states $z_i$ define a higher order state vector $\mathbf{x}$, whose evolution in $\Omega$ is given by the linear equations

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x} \in \Omega, \tag{1.3}$$

where the matrix $A$ contains the matrix $J$ as an upper partition. Our results for the estimation of $\mathbf{x}$ in $\Omega$ will be strongly based on standard theorems for the observability of linear systems [3]. The conditions under which the $N$-th order linear system

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= H'\mathbf{x}, \end{aligned} \tag{1.4}$$

is observable are well known [3]. Observability means knowledge of $\mathbf{y}(\sigma)$, $u(\sigma)$ for $\sigma \leqslant t$ determines $\mathbf{x}(0)$. There are two equivalent necessary and sufficient forms in the case of interest, $A$, $B$, $H$ constant.

(I) The matrix $M(t) = \int_0^t \phi' H H' \phi \, d\tau$ is positive definite (1.5a)

(II) The matrix $R = [H \mid A'H \mid --- \mid A'^{n-1}H]$ has full rank $N$ (1.5b)

where $\phi$ is the transition matrix of $A$. An estimator $\hat{\mathbf{x}}(0)$ for $\mathbf{x}(0)$ at time $t$ can be written

$$\hat{\mathbf{x}}(0) = \phi \left[ M^{-1}(t) \int_0^t \phi' H \left( \mathbf{y} - H'\phi \int_0^\tau \phi' \mathbf{u} \, d\sigma \right) d\tau \right] + \phi \int_0^t \phi^{-1} \mathbf{u} \, d\tau. \tag{1.6}$$

In what follows we shall rely on the known result that for the system of (1.4) if $R$ has full rank then $\hat{\mathbf{x}}(0)$ given by (1.6) exists.

We now illustrate how the above results can be used to estimate the state of a nonlinear system $S$ in a region $\Omega$ about the origin.

EXAMPLE 1. Consider the system $S$ which when expanded in the form (1.2) becomes

$$S: \begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1 + x_2^2, \\ y_1 &= x_2 - x_2^2, \end{aligned} \qquad L: \begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1, \\ y &= x_2. \end{aligned}$$

The system $L$ corresponds to linear terms only of $S$. Note that the observation $y_1$ is also nonlinear. Applying (1.5b) to $L$

$$R = \left[\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right],$$

which has full rank 2. Hence it is possible to estimate the states of $L$ from (1.6). To estimate the states of $S$ define new states

$$x_3 = x_2^2, \qquad x_4 = x_1 x_2, \qquad x_5 = x_1^2$$

and the region $\Omega$ by the condition that all higher order powers and products of $x_i$, i.e., triples, etc., are negligible. Define also a "synthetic observation" in $\Omega$, i.e., $y_2 = y_1^2 \approx x_2^2 = x_3$ in $\Omega$. Then $S$ can be approximated by the system $S^1$ in $\Omega$ given by

$$
\begin{array}{llll}
 & & \dot{x}_3 = 2x_4 , & \\
S^1: & \dot{x}_1 = x_2 , & \dot{x}_4 = x_3 + x_5 , & y_1 = x_2 - x_3 , \\
 & \dot{x}_2 = x_1 + x_3 , & \dot{x}_5 = 2x_4 , & y_2 = x_3 .
\end{array}
$$

Now $S^1$ with 5 states and 2 observations passes the rank test of (1.5b) indicating that an estimator can be generated for the five states from (1.6). The estimator for the states $x_1$ and $x_2$ of $S^1$ will be a linear combination of both $y_1$ and $y_1^2$ and represents an approximation to the states of $S$ in the region $\Omega$.

The procedure is now clear. An autonomous nonlinear system is expanded using (1.2) into a linear part plus a remainder containing higher order terms so that it takes the form of $S^1$ in the example. A region of approximation $\Omega$, is defined and new states consisting of powers and products of original states are defined in $\Omega$ leading to the system $S^1$. Also new synthetic observations are defined in $\Omega$ in terms of powers and products of the old observations.

DEFINITION. The nonlinear system, $S$, is $\Omega$-observable if an exact estimator, $\hat{\mathbf{x}}(0)$, can be found for $S^1$ in $\Omega$.

Hence $S$ is $\Omega$-observable if $S^1$ passes the rank test.

The following example shows that the rank test is only sufficient but not necessary.

EXAMPLE 2.

$$
\begin{array}{llll}
S: & \dot{x}_1 = 0, & L: & \dot{x}_1 = 0, \\
 & y_1 = ax_1^2 + bx_1^3, & & y_1 = 0, \\
 & y_2 = cx_1^2, & & y_2 = 0.
\end{array}
$$

The linearized system $L$ is nonobservable. For $S$ define new states $x_2 = x_1^2$, $x_3 = x_1^3$ and the resulting $A$ matrix remains null indicating failure of the rank condition again. But the estimator

$$
\begin{aligned}
\hat{x}_1 &= [(y_1 - ay_2/c)/b]^{1/3}, \\
\hat{x}_2 &= y_2/c, \\
\hat{x}_3 &= (y_1 - ay_2/c)/b
\end{aligned}
$$

is clearly valid.

One of the major differences between state estimation of a nonlinear system and the observability of a linear system is the way in which a forcing control enters the problem. For the strictly linear plant observability does not depend on the control. The following example shows how the rank test fails for a control $u = 0$ and is satisfied for the control $u = -1$.

EXAMPLE 3.

$$
\begin{aligned}
S:\quad \dot{x}_1 &= x_2 + u, & \qquad L:\quad \dot{x}_1 &= x_2 + u,\\
\dot{x}_2 &= x_2 + x_1^2, & \dot{x}_2 &= x_2\,,\\
y_1 &= x_2 + x_1^2, & y_1 &= x_2\,,\\
y_2 &= x_2^2.
\end{aligned}
$$

For system $L$,

$$
A = \begin{bmatrix} 0 & 1\\ 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 0\\ 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0\\ 1 & 1 \end{bmatrix}.
$$

Since $R$ has rank 1, $L$ is not observable.

For $S$ define $x_3 = x_1^2$, $x_4 = x_1 x_2$, $x_5 = x_2^2$ and $\Omega$ the region where higher order products and powers are neglected.

Then $S$ is approximated in $\Omega$ by a fifth-order linear plant $S^1$ whose $A$ and $H$ matrices are

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0\\ 2u & 0 & 0 & 2 & 0\\ 0 & u & 0 & 1 & 1\\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}; \qquad H = \begin{bmatrix} 0 & 0\\ 1 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1 \end{bmatrix}.
$$

The rank matrix $R$ is given by

$$
R = \left[\begin{array}{cc|cc|cc|cc|cc|cc}
0 & 0 & 2u & 0 & 2u & 0 & 4u & 0 & 8u^2 + 4u & 0 & 16u^2 + 4u & 0\\
1 & 0 & 2 & 0 & 4u + 2 & 0 & 8u + 2 & 0 & 20u + 2 & 0 & 16u^2 + 26u + 2 & 0\\
1 & 0 & 1 & 0 & 2 & 0 & 4u + 2 & 0 & 8u + 2 & 0 & 20u + 2 & 0\\
0 & 0 & 2 & 0 & 4 & 0 & 8 & 0 & 8u + 12 & 0 & 24u + 16 & 0\\
0 & 1 & 0 & 2 & 2 & 4 & 8 & 8 & 24 & 16 & 8u + 60 & 32
\end{array}\right].
$$

For $u = 0$ $R$ has rank 4 and for $u = -1$, $R$ has rank 5 indicating an estimator does exist for $u = -1$.

Our intuition might indicate that if the linearized part of $S$ denoted by $L$ is nonobservable, then the states of $S$ cannot be estimated. Examples 2 and 3 above indicate the contrary. However, for zero forcing control and for estimators restricted to a linear form on the states of $S^1$ as in (1.6) the following theorem applies.

THEOREM. *If a system $S$ is unforced and if its linearized part $L$ is nonobservable then $S^1$, the dynamically linearized approximation to $S$ in $\Omega$, fails the rank test.*

*Proof.* If $L$ takes the form

$$\dot{\mathbf{x}}_1 = A_{11}\mathbf{x}_1, \qquad \mathbf{x}_1 \text{ a } r_1 \text{ vector,}$$

$$\mathbf{y}_1 = H'_{11}\mathbf{x}_1, \qquad \mathbf{y}_1 \text{ a } p_1 \text{ vector,}$$

then by assumption

$$R_1 = [H_{11} \mid A'_{11}H_{11} \mid --- \mid A'^{r_1-1}_{11}H_{11}] \text{ has rank less than } r_1 .$$

The system $S^1$ which creates new states $\mathbf{x}_2$, an $r_2$ vector, from powers and products of $x_1$ takes the form

$$S^1 = \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = A\mathbf{x}.$$

Similarly new synthetic observations $\mathbf{y}_2$, a $p_2$ vector, are created from powers and products of $\mathbf{y}_1$. A linear observation set is created of the form

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} H'_{11} & H'_{12} \\ 0 & H'_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = H'\mathbf{x}.$$

In the system $S^1$ the rank matrix $R_2$ is defined in terms of $A$ and $H$ by (1.5b). We must show that $R_2$ has rank less than $r_1 + r_2$. To show this, note that

$$A'H = \begin{bmatrix} A'_{11}H_{11} & 0 \\ A'_{12}H_{11} + A'_{22}H_{12} & A'_{22}H_{22} \end{bmatrix},$$

and in fact $A'^{(k)}$ takes the form

$$A'^{(k)}H = \begin{bmatrix} A'^{(k)}_{11}H_{11} & 0 \\ (\cdot) & (\cdot) \end{bmatrix}.$$

Hence $R_2$ can be written

$$R_2 = \left[\begin{array}{cc|cc|cc|c|cc} H_{11} & 0 & A'_{11}H_{11} & 0 & A'^{2}_{11}H_{11} & 0 & \cdots & A'^{(r_1+r_2)}_{11}H_{11} & 0 \\ H_{12} & H_{22} & (\cdot) & (\cdot) & (\cdot) & (\cdot) & & (\cdot) & (\cdot) \end{array}\right].$$

But from the Cayley–Hamilton Theorem, there exist constants $\alpha_i$ not all zero such that

$$A'^{(k)}H_{11} = \sum_{i=1}^{r_1} \alpha_i A'^{(i)}H_{11} .$$

Consequently, the upper division of $R_2$ consisting of the first $r_1$ rows spans the space spanned by the columns of $R_1$. This span, by assumption, has dimension less than $r_1$.

Therefore, the projection of the column vectors of $R_2$ on the $r_1$ dimensional subspace of the first $r_1$ rows does not span the entire subspace. It then follows that the column vectors of $R_2$ do not span the entire $r_1 + r_2$ space and hence the rank of $R_2$ is less than $r_1 + r_2$. Q.E.D.

*Remark.* It is known that the rank test is necessary for a linear estimator.

## 2. Synthetic Observations for Stochastic Nonlinear Systems

One of the basic limitations of the Dynamic Linearization approach to nonlinear filtering [1] is the possible lack of observability of the added states which represent the nonlinearities of the system. In Section 1 it was shown how synthetic observations consisting of powers and products of the measured output could be used to estimate the states of the Dynamically Linearized approximating system $S^1$ in $\Omega$. Where measurements are corrupted by white noise, as in the filtering problem, the concept of a synthetic observation must be modified.

Consider the observation vector

$$d\mathbf{r} = H'\mathbf{x}\, dt + R^{1/2}\, d\boldsymbol{\beta}, \tag{2.1}$$

where $\boldsymbol{\beta}$ is a normalized Wiener process which satisfies

$$\begin{aligned} E[d\boldsymbol{\beta}] &= 0, \\ E[d\boldsymbol{\beta}(t)\, d\boldsymbol{\beta}'(\tau)] &= I \mid t - \tau \mid \end{aligned} \tag{2.2}$$

for $I$ the unit diagonal matrix. An attempt to utilize powers and products of $d\mathbf{r}$ would necessarily involve powers and products of $d\boldsymbol{\beta}$. Notice that square terms of $H'\mathbf{x}\, dt$ contain $dt^2$ while square terms of $d\boldsymbol{\beta}$ contain $dt$ on the average. (It can be shown that $d\beta_i(t)\, d\beta_j(t)$ can be replaced identically by its average value, $\delta_{ij}\, dt$). Consequently, in the limit as $dt \to 0$, square and higher order terms in $dr$ depend only on the noise statistics and not on the states.

The concept of synthetic observations adopted from here on consists of a prefilter driven by the measurements whose output and powers and products of its output are used to estimate the states of a Dynamically Linearized system $S^1$ in $\Omega$.

Example 4.

$$\begin{aligned} S\colon\ \dot{x}_1 &= x_1 - x_1^2, \\ y_1 &= x_1 + w_1, \end{aligned}$$

where $w_1 = d\beta/dt$ is white noise.

The prefilter is defined by

$$\dot{x}_2 = \alpha(-x_2 + x_1 + w_1)$$

where $\alpha$ is the inverse time constant of the filter.
New states are defined in $\Omega$ by

$$x_3 = x_2^2, \qquad x_4 = x_1 x_2\,, \qquad x_5 = x_1^2$$

and these propagate according to the rule

$$\begin{aligned}\dot{x}_3 &= 2\alpha(-x_3 + x_4 + x_2 w_1),\\ \dot{x}_4 &= (1-\alpha)\, x_4 + \alpha x_5 + \alpha x_1 w_1\,,\\ \dot{x}_5 &= 2x_5\,.\end{aligned}$$

The additional "synthetic" observations are

$$y_2 = x_2\,, \qquad y_3 = y_2^2 \approx x_3\,,$$

and, being internal to the prefilter, may be regarded as noise free. However, for computational convenience a small white noise will be added to both $y_2$ and $y_3$ thus yielding

$$y_2 = x_2 + w_2; \qquad y_3 = x_3 + w_3\,.$$

The system $S^1$ which approximates $S$ in $\Omega$ can be written in vector form as

$$\begin{aligned} S^1\colon\quad \dot{\mathbf{x}} &= A\mathbf{x} + \alpha B\mathbf{x}w_1 + \alpha\mathbf{c}w_1\,,\\ \mathbf{y} &= H'\mathbf{x} + \mathbf{w}, \end{aligned} \tag{2.4}$$

where

$$\mathbf{w} = \begin{bmatrix} w_1\\ w_2\\ w_3 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 0\\1\\0\\0\\0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0 \end{bmatrix}.$$

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 & 0\\ \alpha & -\alpha & 0 & 0 & 0\\ 0 & 0 & -2\alpha & 2 & 0\\ 0 & 0 & 0 & 1-\alpha & \alpha\\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&2&0&0&0\\1&0&0&0&0\\0&0&0&0&0 \end{bmatrix}.$$

If all noise $\mathbf{w}$ is nulled, then $S^1$ passes the rank test as described in Section 1.

There are several striking new features contributed by the prefilter to the system $S^1$. First of all the term $\alpha B\mathbf{x}w_1$ represents state dependent observation noise which forces the system $S^1$. Also there is a greatly increased computational requirement in that the first-order system $S$ has now become a fifth order system $S^1$. Note that the inverse time constant $\alpha$ has not been specified.

A remaining theoretical point is whether the insertion of the prefilter can defeat the rank test and thereby possibly frustrate the construction of an estimator in the states of $S^1$. The following theorem asserts that it cannot.

THEOREM. *Consider a Dynamically Linearized system $S_1{}^1$ in an approximation region $\Omega$ which passes the rank test. Construct a second system $S_2{}^1$ related to $S_1{}^1$ by inserting a single lag prefilter after each measurement $y_i$ of $S_1{}^1$. In $S_2{}^1$ we observe the synthetic observations out of the prefilter, $z_i$ and the powers and products $z_i z_j \cdots z_k$ which correspond to the observations $y_i$ and $y_i y_j \cdots y_k$ of $S_1{}^1$. In $S_2{}^1$ the observation $y_i$ is retained, but $y_i y_j \cdots y_k$ is no longer available. Then $S_2{}^1$ passes the rank test.*

*Proof.* The processor outputs which we denote by $z_i$ are related to the physical observations $y_i$ by the equation

$$\dot{z}_i/\alpha + z_i = y_i \,. \tag{2.5}$$

By assumption in the system $S_1{}^1$ the $y_i$ and $y_i y_j \cdots y_k$ are observed directly. This implies that the states $x_i$ and the extra states $x_i x_j \cdots x_k$ created by dynamic linearization are known from the satisfaction of the rank test. In $S_2{}^1$ we observe directly $y_i$, $z_i$ and powers and products of the form $z_i z_j \cdots z_k$. From Eq. (2.5) $\dot{z}_i$ is then determined as a linear combination of observables. Since $S_2{}^1$ represents a continuous system all derivatives of quantities such as $z_i z_j \cdots z_k$ are linear combinations of their past values. Also terms such as $z_i z_j \cdots z_k$ can be represented as linear combinations of derivatives of observable quantities $z_1 z_m \cdots z_m$. We must show the following: (1) the original states $x_i$, $x_i x_j \cdots x_k$ are linear combinations of $y_i$, $z_i$ and $z_i z_j \cdots z_k$, 2) new states created by the prefilter $z_i x_j x_k \cdots x_1$ are linear combinations of these same observable quantities.

To show (1) above note that $y_i y_j \cdots y_k$ is a linear combination of powers and products of the derivatives of the $z_i$ which were shown above to be linearly related to observable quantities. But from the rank test on $S_1{}^1$ the states $x_i$ and $x_i x_j x_k$ depend linearly on $y_i$ and $y_i y_j \cdots y_k$. To show (2) again use the rank test on $S_1{}^1$ to replace $x_i x_j x_k$ by a linear combination of the $y_i y_j$. Then $z_i x_j x_k \cdots x_1$ is a linear combination of $z_i y_j y_k \cdots y_1$. From (2.5) this may be expressed in terms of a linear combination of terms in $z_i z_j z_k$ which were shown above to be linearly related to observables. Hence all states are linearly related to present and past values of $y_i$, $z_i$, $z_i z_j \cdots z_k$ and hence since the rank test is necessary in a linear estimator it must be satisfied.

## 3. Conclusions

The familiar rank test for the observability of linear systems can be used as a sufficient but not necessary condition for the $\Omega$-observability of a nonlinear system. $\Omega$-observability depends on the utilization of "synthetic observations" which are constructed from the real physical measurements, and also on the nature of any forcing functions. The concept is expected to be of use in nonlinear filtering theory.

## References

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